MATH 1101 FINAL EXAM SOLUTIONS

Dec 20, 2005

1. (15 pts) Evaluate the following. Be sure to justify your answers.

   (a) $\lim_{x \to 0^+} \left(\frac{1}{x^2}\right)^x$

   First write

   $$\left(\frac{1}{x^2}\right)^x = \left(e^{\ln(1/x^2)}\right)^x = e^{-2\ln(x)x}$$

   Now

   $$\lim_{x \to 0^+} -2\ln(x)x = -2 \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

   As $x \to 0^+$, $\ln(x) \to \infty$ and $1/x \to \infty$. Both $\ln(x)$ and $1/x$ are differentiable for $x > 0$. $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$ which is not 0 near $x = 0$. So we can use L'Hospital's rule:

   $$\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} x = 0$$

   We know $f(x) = e^x$ is continuous, so

   $$\lim_{x \to 0^+} \ln(x)x = 0 \implies \lim_{x \to 0^+} e^{-2\ln(x)x} = 1$$

   (b) $\lim_{t \to 0} \frac{\cos(t) - 1}{t^2}$

   As $t \to 0$, $\cos(t) - 1 \to 0$ and $t^2 \to 0$, so this is another candidate for l'Hospital's rule.

   $$\frac{d}{dt}(\cos(t) - 1) = -\sin(t), \qquad \frac{d}{dt}t^2 = 2t$$

   so both the numerator and the denominator are differentiable near 0. Except at 0, $2t \neq 0$ near 0. So l'Hospital's rule can indeed be used:

   $$\lim_{t \to 0} \frac{\cos(t) - 1}{t^2} = \lim_{t \to 0} \frac{-\sin(t)}{2t} = -\frac{1}{2} \lim_{t \to 0} \frac{\sin(t)}{t} = -\frac{1}{2}$$

   as we proved in class that

   $$\lim_{t \to 0} \frac{\sin(t)}{t} = 1$$

   (c)

   $$\begin{aligned}\frac{d}{dx} \ln \sqrt[x]{\sin^2(x)\cosh(x)} &= \frac{d}{dx}\left[\frac{1}{x}\left(2\ln(\sin x) + \ln(\cosh x)\right)\right] \\ &= -\frac{1}{x^2}\left(2\ln(\sin x) + \ln(\cosh x)\right) + \frac{1}{x}\left(2\frac{\cos x}{\sin x} + \frac{\sinh x}{\cosh x}\right) \\ &= -\frac{1}{x^2}\left(2\ln(\sin x) + \ln(\cosh x)\right) + \frac{1}{x}\left(2\cot(x) + \tanh(x)\right)\end{aligned}$$

2. (30 pts)

   (a) True or false: A strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$ always has an inverse? If true, prove it, if not, give a counterexample.

   If f is strictly decreasing then it can't take on the same value twice. So it's always one-to-one. But it could fail to be onto. For example, $f(x) = e^{-x}$ takes on only positive values. So as a function $\mathbb{R} \to \mathbb{R}$ it wouldn't have an inverse. What would $f^{-1}(-1)$ be? We could fix this problem by restricting the codomain of f to it its range, which would make f onto and therefore invertible.

(b) Show that

$$\log_6\left(\frac{99}{20}\right) = 2 - 4\log_6(2) - \log_6(5) + \log_6(11)$$

$$\begin{aligned}\log_6\left(\frac{99}{20}\right) &= \log_6\left(\frac{3^2 \cdot 11}{2^2 \cdot 5}\right) = \log_6\left(\frac{2^2 \cdot 3^2 \cdot 11}{2^4 \cdot 5}\right)\\ &= \log_6\left(\frac{6^2 \cdot 11}{2^4 \cdot 5}\right)\\ &= 2\log_6(6) + \log_6(11) - 4\log_6(2) - \log_6(5)\\ &= 2 - 4\log_6(2) - \log_6(5) + \log_6(11)\end{aligned}$$

(c) Let $f(x) = 1/x^2$. Use the definition of the derivative to find $f'(x)$.

$$\begin{aligned}f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{t\to x}\frac{\frac{1}{t^2}-\frac{1}{x^2}}{t-x} = \lim_{t\to x}\frac{x^2-t^2}{t^2x^2(t-x)}\\ &= \lim_{t\to x}\frac{(x-t)(x+t)}{t^2x^2(t-x)} = \lim_{t\to x} -\frac{x+t}{t^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}\end{aligned}$$

(d) Is $\operatorname{arcsinh}(\sinh(x)) = x$ for all $x \in \mathbb{R}$? Justify your answer.

We know sinh is one-to-one and onto from class, so it has an inverse. This inverse is arcsinh. Hence $\operatorname{arcsinh}(\sinh(x)) = x$ for all $x \in \mathbb{R}$.

(e) Use a linear approximation to $f(x) = \sqrt{x}$ at $x_0 = 100$ to approximate $\sqrt{101}$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

We have $f'(x) = (1/2)x^{-1/2}$ and $x_0 = 100$, so

$$\sqrt{(101)} = f(101) \approx \sqrt{100} + \frac{1}{2\sqrt{100}}(101-100) = 10 + \frac{1}{20} = 10.05$$

(f) Use the Fundamental Theorem of Calculus to find

$$\int_{-\pi/2}^{\pi/2} \cos(2x)\,dx$$

Plot the graph of $f(x) = \cos(2x)$. Does your result for the integral make sense?

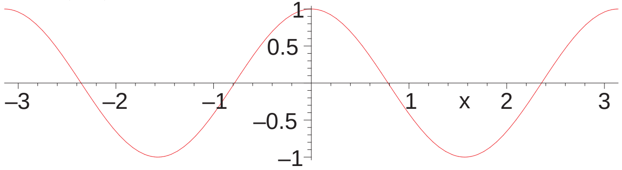
Since

$$\frac{d}{dx}\sin(2x) = 2\cos(2x) \implies \frac{d}{dx}\frac{\sin(2x)}{2} = \cos(2x)$$

the FTC says

$$\int_{-\pi/2}^{\pi/2} \cos(2x)\,dx = \left.\frac{\sin(2x)}{2}\right|_{-\pi/2}^{\pi/2} = \frac{\sin(\pi)}{2} - \frac{\sin(-\pi)}{2} = 0$$

Here is the graph of $\cos(2x)$:



By symmetry, the area between the graph and the x-axis between $-\pi/2$ and $-\pi/4$ is the same as between $-\pi/4$ and 0, only one is below the x-axis while the other is above, so they cancel. The same thing happens on the positive side of the x-axis. So the above result makes sense.

3. (10 pts) Find the equation of the tangent line to the curve

$$x^y = x^3 - 3xy - e^3 + 1$$

at the point $(e, 0)$. As you differentiate, keep in mind that y is a function of x and treat x^y accordingly. (Hint: Is x^y a power function? Is it an exponential function? Is it neither?)

We use implicit differentiation to differentiate the equation:

$$\begin{aligned}
\frac{d}{dx}x^y &= \frac{d}{dx}(x^3 - 3xy - e^3 + 1) \\
\frac{d}{dx}e^{y\ln(x)} &= 3x^2 - 3y - 3x\frac{dy}{dx} \\
e^{y\ln(x)}\left(\frac{dy}{dx}\ln(x) + \frac{y}{x}\right) &= 3x^2 - 3y - 3x\frac{dy}{dx} \\
x^y\left(\frac{dy}{dx}\ln(x) + \frac{y}{x}\right) &= 3x^2 - 3y - 3x\frac{dy}{dx}
\end{aligned}$$

When $x = e$ and $y = 0$

$$\begin{aligned}
e^0\left(\frac{dy}{dx}\ln(e) + \frac{0}{e}\right) &= 3e^2 - 3(0) - 3e\frac{dy}{dx} \\
\frac{dy}{dx} &= 3e^2 + 3e\frac{dy}{dx} \\
(1+3e)\frac{dy}{dx} &= 3e^2 \\
\frac{dy}{dx} &= \frac{3e^2}{1+3e}
\end{aligned}$$

This is the slope of the tangent line. The line also goes through $(e, 0)$, so if its equation is $y = mx + b$, then

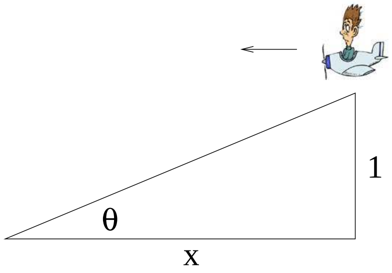
$$0 = \frac{3e^2}{1+3e}e + b \implies b = -\frac{3e^3}{1+3e}$$

and the equation of the line is

$$y = \frac{3e^2}{1+3e}x - \frac{3e^3}{1+3e}$$

4. (10 pts) The Washington Metropolitan Air Defense Identification Zone (ADIZ) uses a visual warning laser to alert pilots who enter the ADIZ without proper clearance. An alternating red and green laser light is aimed at the cockpit of the unauthorized airplane. Suppose that a small airplane is flying straight and level at an altitude of 6100 ft toward the ground station where the laser is installed. The plane's speed is 100 knots (nautical miles/hour). At what

rate in radians/minute does the laser need to turn to stay aimed at the airplane when the plane is 10 nautical miles from the laser? Simplify your answer as much as you can. You may use your result from Problem 2e or just leave a $\sqrt{\ }$ in it. FYI, $1\,\mathrm{nm} \approx 6100\,\mathrm{ft}$.



As you can see from the picture,

$$\tan(\theta) = \frac{1}{x}$$

Differentiate both sides w.r.t. t:

$$\sec^2(\theta)\frac{d\theta}{dt} = -\frac{1}{x^2}\frac{dx}{dt} \implies \frac{d\theta}{dt} = -\frac{1}{x^2}\frac{dx}{dt}\cos^2(\theta)$$

When $x = 10$ in the picture above,

$$\cos^2(\theta) = \left(\frac{10}{\sqrt{101}}\right)^2 = \frac{100}{101}$$

We know $\frac{dx}{dt} = -100$ nm/h (the plane is flying toward the laser, so its distance to the laser is decreasing). Hence

$$\frac{d\theta}{dt} = -\frac{1\,\mathrm{nm}}{(10\,\mathrm{nm})^2}\left(-100\,\frac{\mathrm{nm}}{\mathrm{h}}\right)\frac{100}{101} = \frac{100}{101}\frac{1}{\mathrm{h}} = \frac{100}{101}\frac{1}{60\,\mathrm{min}} = \frac{10}{606}\frac{1}{\mathrm{min}}$$

This is now in radians/min.

As you can see, we didn't actually need to know $\sqrt{101}$ for this.

5. (10 pts) As the annual Elves vs. Reindeer hockey game is drawing near, Santa realizes that the old ice rink is no longer holding up and he needs to build a new one. Making the ice itself is no problem at the North Pole, but with temperatures hovering around −110°F with windchill, he needs to cover it too. He scrapes together the \$31415 left in his account after buying all those toys, and decides on a tent to cover the rink. The tent has the shape of half a cylinder. Its ends are made of corrugated aluminum and the side is of canvas. Fortunately, the Home Depot's winter 3-for-the-price-of-1 sale includes corrugated aluminum at \$4 for $3\,\mathrm{ft}^2$ and canvas at \$2 for $3\,\mathrm{ft}^2$. Santa wants to maximize the volume of the tent. What are the dimensions of the biggest tent he can afford to build?

Let r be the radius of the half-cylinder and x be its length. The volume of the tent is

$$V = \frac{\pi}{2}r^2x$$

This depends on two variables, so we need to eliminate one.

The two ends combined are exactly a circle and their area is πr^2. The top of the tent has area $\pi r x$. Obviously, Santa needs to spend all his money if he wants to maximize the volume of the tent. Denote Santa's stash of cash by M

$$\begin{aligned}\frac{4}{3}\pi r^2 + \frac{2}{3}\pi r x &= M\\ \frac{2}{3}\pi r x &= M - \frac{4}{3}\pi r^2\\ x &= \frac{3M}{2\pi r} - 2r\end{aligned}$$

So

$$V(r) = \frac{\pi}{2} r^2 \left(\frac{3M}{2\pi r} - 2r\right) = \frac{3Mr}{4} - \pi r^3$$

To maximize V, we find where $V'(r) = 0$.

$$V'(r) = \frac{3M}{4} - 3\pi r^2 = 0 \implies r = \sqrt{\frac{M}{4\pi}} = \sqrt{\frac{31415}{4\pi}} \approx \sqrt{\frac{10000}{4}} = \frac{100}{2} = 50$$

To be sure this is a local maximum, we'll do the 2nd derivative test:

$$V''(r) = -6\pi r < 0$$

so indeed we have found a maximum. The dimensions of this tent are

$$\begin{aligned}r &= 50\,\text{ft}\\ x &= \frac{3M}{2\pi r} - 2r \approx \frac{3(31415)}{100\pi} - 100 \approx 300 - 100 = 200\,\text{ft}\end{aligned}$$

6. (10 pts) Use Rolle's Theorem and mathematical induction to prove that a polynomial of degree n has at most n real roots.

   Note to Math 1101 students in spring 2006: we didn't cover induction this semester, so we will do the problem without it.

   First notice that the graph of a polynomial of degree 1 is a straight line, so it can't cross the x-axis more than once, hence such a polynomial has at most one root.

   Now, let $p(x)$ be a polynomial of degree 2. Suppose p has more than 2 roots. Let $a_1 < a_2 < a_3$ be three of those roots. Then $p(a_1) = p(a_2) = p(a_3)$. Since p is a polynomial, it is continuous and differentiable everywhere. So Rolle's Theorem says there exist b_1 and b_2 such that $a_1 < b_1 < a_2 < b_2 < a_3$ and $p'(b_1) = p'(b_2) = 0$. But p' is a polynomial of degree 1, so it can't have more than one root. Therefore p can't have more than 2 roots.

   Now, let $p(x)$ be a polynomial of degree 3. Suppose p has more than 3 roots. Let $a_1 < a_2 < a_3 < a_4$ be four of those roots. Then $p(a_1) = p(a_2) = p(a_3) = p(a_4)$. Since p is a polynomial, it is continuous and differentiable everywhere. So Rolle's Theorem says there exist b_1, b_2, and b_3 such that $a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < a_4$ and $p'(b_1) = p'(b_2) = p'(b_3) = 0$. But p' is a polynomial of degree 2, so it can't have more than two roots. Therefore p can't have more than 3 roots.

   Continue this way to see that a polynomial of degree n can't have more than n roots.

7. (20 pts) **Extra credit problem.** We will prove Cauchy's Mean Value Theorem, which is a generalized version of the MVT. It says that if f and g are functions that are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for any $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}$$

(a) Our first attempt to prove this might be as follows. Given the above conditions on f, the MVT says there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Similarly, the MVT says that there exists a $c \in (a, b)$ such that

$$g'(c) = \frac{g(a) - g(b)}{a - b}$$

Now divide these last two equations to get

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(a)-f(b)}{a-b}}{\frac{g(a)-g(b)}{a-b}} = \frac{f(a) - f(b)}{g(a) - g(b)}$$

Find the mistake in this argument.

The MVT says there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

It also says there exists a $d \in (a, b)$ such that

$$g'(d) = \frac{g(a) - g(b)}{a - b}$$

But there is no reason to believe $c = d$.

(b) Recall how the proof of the MVT went. We let

$$y = \frac{f(a) - f(b)}{a - b}(x - a) + f(a),$$

which is the equation of the secant line to f between a and b. Then we let

$$h(x) = f(x) - y = f(x) - \frac{f(a) - f(b)}{a - b}(x - a) - f(a)$$

and used Rolle's Theorem on h to claim that c exists. Now we will let

$$h(x) = f(x) - \frac{f(a) - f(b)}{g(a) - g(b)}(g(x) - g(a)) - f(a)$$

instead. Verify that h satisfies the three conditions of Rolle's Theorem.

$$\begin{aligned} h(a) &= f(a) - \frac{f(a) - f(b)}{g(a) - g(b)}(g(a) - g(a)) - f(a) = 0 \\ h(b) &= f(b) - \frac{f(a) - f(b)}{g(a) - g(b)}(g(b) - g(a)) - f(a) \\ &= f(b) + (f(a) - f(b)) - f(a) = 0 \end{aligned}$$

Since f and g are continuous on $[a, b]$, so is h, unless $g(a) - g(b) = 0$, in which case we would have division by 0. But if $g(a) = g(b)$, then by Rolle's Theorem, there would be a point $c \in (a, b)$ such that $g'(c) = 0$. But we know $g'(x) \neq 0$ anywhere on (a, b). So we can be certain $g(a) \neq g(b)$.
Since f and g are differentiable on (a, b)

$$h'(x) = f'(x) - \frac{f(a) - f(b)}{g(a) - g(b)}g'(x)$$

showing h is differentiable on (a, b) unless $g(a) = g(b)$, but we already ruled out this possibility.
So Rolle's Theorem applies.

(c) Now use Rolle's Theorem on h to find a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}$$

Rolle's Theorem says there is a point $c \in (a, b)$ such that

$$0 = h'(c) = f'(c) - \frac{f(a) - f(b)}{g(a) - g(b)} g'(c)$$

$$f'(c) = \frac{f(a) - f(b)}{g(a) - g(b)} g'(c)$$

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}$$