Math 1101 Exam 2 Solutions

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1. (15 pts) Find the following limits. If the limit is not a real number or $\pm\infty$, say so. Be sure to justify your answers.

   (a) $\lim_{x\to 0} \frac{\sqrt{3x+2}-\sqrt{x+2}}{x}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3x+2}-\sqrt{x+2}}{x} &= \lim_{x\to 0} \frac{(\sqrt{3x+2}-\sqrt{x+2})(\sqrt{3x+2}+\sqrt{x+2})}{x(\sqrt{3x+2}+\sqrt{x+2})} \\
&= \lim_{x\to 0} \frac{(3x+2)-(x+2)}{x(\sqrt{3x+2}+\sqrt{x+2})} = \lim_{x\to 0} \frac{2x}{x(\sqrt{3x+2}+\sqrt{x+2})} \\
&= \lim_{x\to 0} \frac{2}{\sqrt{3x+2}+\sqrt{x+2}} = \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{1}{\sqrt{2}}
\end{aligned}$$

   (b) $\lim_{y\to -\infty} \frac{4y^3-3y+10}{5+2y-8y^3}$

   As $x \to -\infty$, the cubic terms in the numerator and the denominator become dominant over the lower degree terms. So

$$\lim_{y\to -\infty} \frac{4y^3-3y+10}{5+2y-8y^3} = \lim_{y\to -\infty} \frac{4y^3}{-8y^3} = \lim_{y\to -\infty} \frac{4}{-8} = -\frac{1}{2}$$

   (c) $\lim_{x\to 3^-} \frac{x-3}{|x-3|}$

   If $x \to 3^-$, then $x < 3$, so $|x-3| = -(x-3)$. Hence

$$\lim_{x\to 3^-} \frac{x-3}{|x-3|} = \lim_{x\to 3^-} \frac{x-3}{-(x-3)} = \lim_{x\to 3^-} -1 = -1$$

2. (15 pts)

   (a) Define what it means for a function $f : S \to T$ to be one-to-one.

   f is one-to-one if it is true that for any $s_1, s_2 \in S$, if $s_1 \neq s_2$, then $f(s_1) \neq f(s_2)$.

   (b) Can an even function $f : \mathbb{R} \to \mathbb{R}$ be one-to-one? Why or why not?

   No. If f is even, then $f(1) = f(-1)$, but then f takes on the same value for two different inputs and therefore cannot be one-to-one.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Prove that f must be one-to-one.

   Let $x_1 \neq x_2 \in \mathbb{R}$. Then either $x_1 < x_2$ or $x_2 < x_1$. In the first case, $f(x_1) > f(x_2)$, and the in the second, $f(x_2) > f(x_1)$. In any case, $f(x_1) \neq f(x_2)$. So f is one-to-one.

3. (10 pts) Let $f(x) = e^{5x-3}$.

   (a) Find the inverse of f.

$$y = e^{5x-3}$$
$$\ln(y) = 5x - 3$$
$$\frac{\ln(y)+3}{5} = x \qquad \Longrightarrow \; f^{-1}(x) = \frac{\ln(x)+3}{5}$$

(b) Check your answer by finding $f(f^{-1}(x))$. What should $f(f^{-1}(x))$ be?

$$f(f^{-1}(x)) = e^{5\frac{\ln(x)+3}{5}-3} = e^{\ln(x)} = x$$

which is exactly what $f(f^{-1}(x))$ should be. So the answer to part (a) is likely to be correct.

4. (10 pts) *Habañero*[1] *salsa dating.* On a secret archeological expedition in southern Mexico, Indiana Jones finds a mummy holding a tightly closed clay jar. The content of the jar is identified in the lab as habañero salsa. It is believed that the fieriness of salsafied habañero chilies declines exponentially with time. Fresh habañero salsa rates at 100000 Scoville Units (SU). An authentic 1000-year old sample comes in at 25000 SU. (Habañeros are some of spiciest chillies in the world, and Scoville Units are the usual measures of spicyness. But none of this is important for solving this problem.)

(a) Find $f(t)$, the heat of a t-year old habañero salsa in SU.

Since the next part asks for the half-life, we will use the half-life equation:

$$f(t) = 100000\left(\frac{1}{2}\right)^{t/h}$$

where h is the half-life. To find h, we use

$$25000 = f(1000) = 100000\left(\frac{1}{2}\right)^{1000/h}$$
$$\frac{25000}{100000} = \left(\frac{1}{2}\right)^{1000/h}$$
$$\frac{1}{4} = \left(\frac{1}{2}\right)^{1000/h}$$
$$2 = \frac{1000}{h}$$
$$h = 500$$

So

$$f(t) = 100000\left(\frac{1}{2}\right)^{t/500}$$

(b) What is the half-life of the fire in habañero salsa?

From the previous part, the half-life is 500 years.

(c) Dr. Jones field tests his habañero salsa with a tortilla chip and rates it at 5000 SU. How old is the mummy? (You may leave logs in your answer.)

$$
\begin{aligned}
5000 &= 100000 \left(\frac{1}{2}\right)^{t/500} \\
\frac{5000}{100000} &= \left(\frac{1}{2}\right)^{t/500} \\
\frac{1}{20} &= \left(\frac{1}{2}\right)^{t/500} \\
\ln\left(\frac{1}{20}\right) &= \frac{t}{500} \ln\left(\frac{1}{2}\right) \\
500\frac{-\ln(20)}{-\ln(2)} &= t
\end{aligned}
$$

So the mummy is $500 \ln(20)/\ln(2)$ years old.

5. (15pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.

   Let $f : S \to T$ and $g : T \to U$ be onto functions. Show that $g \circ f$ is an onto function.

   Let u be any element in U. Since g is onto, there exists some $t \in T$ such that $g(t) = u$. Since f is onto, there exists some $s \in S$ such that $f(s) = t$. Hence

$$g \circ f(s) = g(f(s)) = g(t) = u$$

   We have just shown that for any $u \in U$, there exists a $s \in S$ such that $g \circ f(s) = u$. This is exactly what it means for $g \circ f$ to be onto.